Dimensional Analysis

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Example 1 (Period of a Pendulum). *Consider a simple pendulum, meaning an object of mass m attached to the end of an inextensible string of length ℓ, immersed in Earth's gravitational field with gravitational acceleration g. The other end of the string is fixed to the ceiling. Is it possible to determine how the pendulum's period depends on the other quantities that define the system without using Newton's laws?*

Of course, the answer is yes! This method is called *dimensional analysis*. Here's how it works:

1. **Identify the quantities that may influence the one we wish to find**. In this case, they are:

   - the mass m, whose dimension is denoted as $[M]$;
   - the magnitude of gravitational acceleration g, with dimension $[g] = \frac{[L]}{[T^2]}$;
   - the length ℓ of the string, with dimension $[L]$.

   Dimensions are represented by uppercase letters in square brackets, or simply by units of measurement (kg, m, s...). Length is denoted by $[L]$, time by $[T]$, mass by $[M]$, temperature by $[\theta]$, and electric charge by $[Q]$.

2. **Express the target quantity as a product of the identified quantities, each raised to an unknown exponent**. In this case, the period of the pendulum is written as
$$P = C\, m^\alpha g^\beta \ell^\gamma,$$
   where C is a dimensionless constant and α, β, and γ are the exponents to be determined.

3. **Translate the equation into a relation among dimensions of the fundamental quantities**. In this case:
$$[T] = [M]^\alpha \left(\frac{[L]}{[T]^2}\right)^\beta [L]^\gamma,$$
   where the constant C has been omitted, as it is dimensionless.

4. **Equate the dimensions on both sides of the equation**. This gives three equations, one for each fundamental dimension:
$$\begin{cases} 1 = -2\beta, \\ 0 = \alpha, \\ 0 = \beta + \gamma. \end{cases}$$

5. **Solve the resulting system of equations**. The solution is $\alpha = 0$, $\beta = -1/2$, and $\gamma = 1/2$.

6. **Substitute the values of the exponents into the original expression**. This yields:

$$P = C\, m^{\alpha} g^{\beta} \ell^{\gamma} = C\, m^{0}\, g^{-1/2}\, \ell^{1/2} = C\sqrt{\frac{\ell}{g}}.$$

The problem is now solved!

Problem 1 (**Dimensions of Derived Quantities**)**.** Determine the dimensions of the following physical quantities in terms of the fundamental dimensions $[L]$, $[T]$, $[M]$, $[\Theta]$, and $[Q]$:

- energy,
- angular momentum,
- electric field,
- magnetic field,
- specific heat capacity,
- thermal conductivity.

Problem 2 (**Wave Speed on a Tensed String**)**.** A homogeneous string of mass m and length l is under a tension τ. What is the speed at which transverse waves propagate along the string?

Problem 3 (**Falling Object**)**.** Given a height h and gravitational acceleration g, how can one construct a velocity using these quantities? How can one construct a time?

Problem 4 (**Cooking Time**)**.** Assume that the time required to cook a homogeneous slice of meat follows the relation

$$t = A\,\frac{c}{k}\,\rho^{\alpha}\, m^{\frac{2}{3}},$$

where A is a dimensionless constant, c is the specific heat capacity of the meat, k is its thermal conductivity, ρ is its density, and m is its mass. What is the value of the exponent α?

Problem 5 (**Atomic Fireball** — from the Physics Team Competition)**.** When a nuclear bomb explodes, a fireball is formed and expands rapidly. The fireball of the first atomic bomb had a radius of 80 m after 0.006 s from the explosion. Assuming that the fireball's expansion depends only on the energy released and on the air density, what was its radius after 0.016 s?

Problem 6 (**Power Radiated by an Accelerated Charge**)**.** The electromagnetic power emitted by an accelerating charged particle depends on its electric charge q, its acceleration a, the vacuum permittivity constant ϵ_0, and the speed of light in vacuum c. Determine the functional dependence of the radiated power on these quantities.

Problem 7 (Solar Estimates). Model the Sun as a stationary sphere of mass M and radius R, made of monoatomic hydrogen at absolute temperature T. The total energy of the Sun can be written as

$$E = W + U,$$

where W is thermal energy and U is gravitational potential energy. The latter can be expressed as

$$U = A\,GM^{\alpha}R^{\beta},$$

where G is the universal gravitational constant and A is a dimensionless constant of order unity. The thermal energy is given by

$$W = B\,k_B T^{\gamma},$$

where k_B is the Boltzmann constant and B is a dimensionless constant.

1. Determine the values of α, β, and γ.
2. Given that $W = -E$, express the absolute temperature T of the Sun as a function of M and R.

Problem 8 (Flying Scaling). An aircraft is able to fly at constant altitude while moving at speed u with respect to the ground.

- Keeping all other parameters fixed, at what speed must an aircraft with double the size in all spatial dimensions (x, y, and z) fly to maintain the same altitude as the original?
- If P is the power produced by the engines of the original aircraft, how much power must the engines of the scaled-up aircraft generate to keep it at the same altitude?

Solutions

Solution 1 (Dimensions of Derived Quantities). The answers are:

- energy $\frac{[M][L]^2}{[T]^2}$,
- angular momentum $\frac{[M][L]^2}{[T]}$,
- electric field $\frac{[M][L]}{[Q][T]^2}$,
- magnetic field $\frac{[M]}{[Q][T]}$,
- specific heat capacity $\frac{[L]^2}{[T]^2[\Theta]}$,
- thermal conductivity $\frac{[M][L]}{[T]^3[\Theta]}$.

Solution 2 (Wave Speed on a Tensed String).

$$v = C\sqrt{\frac{\tau l}{m}}$$

Solution 3 (**Falling Object**).

$$v = C\,\sqrt{gh}, \qquad\qquad t = C\,\sqrt{\frac{h}{g}}.$$

Solution 4 (**Cooking Time**).

$$\alpha = \frac{1}{3}.$$

Solution 5 (**Atomic Fireball**). The problem states that the fireball radius depends only on the energy E released and the air density ρ. One more quantity must be included to describe the evolution over time: the time t itself.

Let us seek a combination of t, E, and ρ that has the dimension of a length:

$$R = C\,t^{\alpha} E^{\beta} \rho^{\gamma},$$

where α, β, and γ are real numbers. Using dimensional analysis:

$$[L] = [T]^{\alpha} \left(\frac{[M][L]^2}{[T]^2} \right)^{\beta} \left(\frac{[M]}{[L]^3} \right)^{\gamma}.$$

Matching the dimensions on both sides yields the system:

$$\begin{cases} 1 = 2\beta - 3\gamma, \\ 0 = \alpha - 2\beta, \\ 0 = \beta + \gamma, \end{cases}$$

with the solution: $\alpha = 2/5$, $\beta = 1/5$, $\gamma = -1/5$.

Thus:

$$R(t) = C\,t^{2/5} E^{1/5} \rho^{-1/5}, \quad \text{so} \quad R(t_2) = R(t_1) \left(\frac{t_2}{t_1} \right)^{2/5} \approx 106\,\mathrm{m},$$

where $t_1 = 0.006\,\mathrm{s}$ and $t_2 = 0.016\,\mathrm{s}$.

Solution 6 (**Power Radiated by an Accelerated Charge**).

$$P = C\,\frac{q^2 a^2}{\epsilon_0\, c^3}.$$

Solution 7 (**Solar Estimates**).

$$\alpha = 2, \qquad \beta = -1, \qquad \gamma = 1.$$

$$T = -\frac{A\,GM^2}{2Bk_B R},$$

The temperature is not negative because $A < 0$. Also, note that since thermal energy is an extensive quantity, B must be proportional to the number of atoms in the Sun.

Solution 8 (**Flying Scaling**). Dimensional analysis shows that the lift force opposing gravity is given by:

$$F = C\,\rho_{\mathrm{air}} A v^2,$$

where A is an area.

1. For the original aircraft:

$$C\,\rho_{\text{air}} A u^2 \sim mg,$$

and for the enlarged model:

$$C\,\rho_{\text{air}} A' u'^2 \sim m' g.$$

Dividing the two equations:

$$\left(\frac{u'}{u}\right)^2 = \frac{Am'}{A'm} = \frac{A(2^3 m)}{(2^2 A)m} = 2,$$

hence:

$$u' = \sqrt{2}\,u.$$

2. The required engine power, proportional to $F\,u$, scales as:

$$P' = 2^{7/2} P.$$